9.1 Periodic Functions and Fourier Series

recall Taylor series: find out how $f(x)$ is made up of the building blocks x^n

for example, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

to build e^x, we need one part of 1, one part of x, $\frac{1}{2!}$ parts of x^2, etc

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

equal parts of x^n

these x^n are called the <u>basis functions</u>

(generalization of, for example, $\vec{i}, \vec{j}, \vec{k}$ in $\mathbb{R}^3$)

for Taylor, $f(x)$ needs to infinitely differentiable at $x=a$

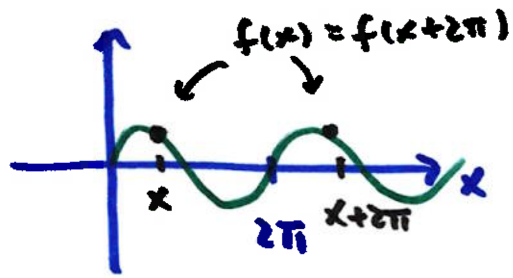
Fourier series have the basic idea: individual "parts" of $f(x)$

but not using x^n as basis functions, but use $\cos(nx)$ and $\sin(nx)$

as long as $f(x)$ is periodic and piecewise smooth

periodic: $f(x) = f(x+T)$ → $f(x)$ is periodic w/ period T

$\sin(x) = \sin(x+2\pi)$ → $\sin(x)$ has period 2π



$\sin(x) = \sin(x+2\pi) = \sin(x+4\pi) = \sin(x+k\cdot 2\pi)$

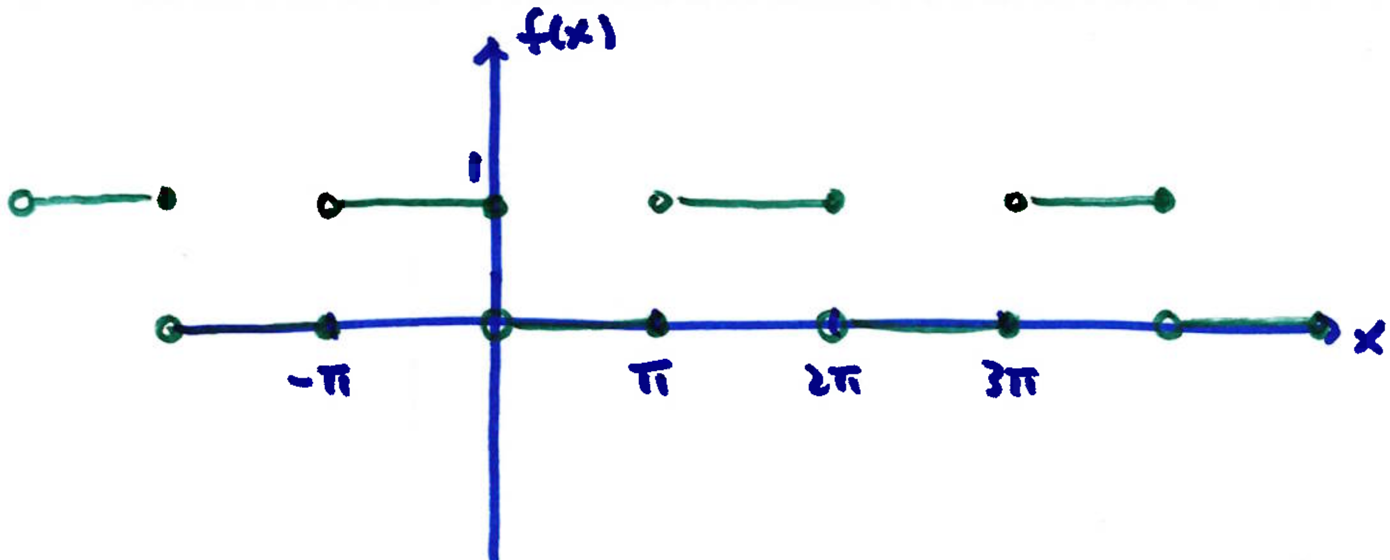
$k = 1, 2, 3, \ldots$

$f(x) = f(x+T) = f(x+kT)$

shortest such period is the fundamental period what is the fundamental period of $\sin(3x)$

$$T = \frac{2\pi}{3}$$

this function is periodic with period 2π and piecewise smooth



expressed as $f(x) = \begin{cases} 1 & -\pi < x \leq 0 \\ 0 & 0 < x \leq \pi \end{cases}$ period is 2π

can be expressed as sum of $\cos(nx)$ and $\sin(nx)$

Fourier series for $f(x)$ w/ period 2π defined on $-\pi < x < \pi$ is

$$f(x) = \frac{1}{2}\underbrace{a_0}_{a_0 \cdot 1 = a_0 \cdot \cos(0x)} + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + \cdots$$

$$+ b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x) + \cdots$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

a_n, b_n are coefficients of basis functions

(like $\frac{f^{(n)}(a)}{n!}$ in Taylor series)

How to find a_n and b_n ?

3 important properties of cosine and sine

$$\int_{-\pi}^{\pi} \cos(\alpha x)\cos(\beta x)\,dx = \begin{cases} \pi & \text{if } \alpha = \beta \neq 0 \\ 2\pi & \text{if } \alpha = \beta = 0 \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(\alpha x)\sin(\beta x)\,dx = \begin{cases} \pi & \text{if } \alpha=\beta\neq 0 \\ 0 & \text{if } \alpha=\beta=0 \\ 0 & \text{if } \alpha\neq\beta \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(\alpha x)\sin(\beta x)\,dx = 0$$

cosines and sines
are mutually orthogonal

$$f(x) = \frac{1}{2}a_0 + a_1\cos(x) + a_2\cos(2x) + \cdots + a_n\cos(nx) + \cdots$$
$$+\, b_1\sin(x) + b_2\sin(2x) + \cdots + b_n\sin(nx) + \cdots$$

multiply both sides by $\cos(nx)$ and integrate over $-\pi < x < \pi$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \cancelto{\text{if } n\neq 0}{\int_{-\pi}^{\pi}\frac{1}{2}a_0\cos(0x)\cos(nx)\,dx} + \cancelto{\text{if } n\neq 1}{\int_{-\pi}^{\pi} a_1\cos(x)\cos(nx)\,dx}$$
$$+ \cdots + \cancelto{0}{\int_{-\pi}^{\pi} b_1\sin(x)\cos(nx)\,dx} + \cancelto{0}{\int_{-\pi}^{\pi} b_2\sin(2x)\cos(nx)\,dx}$$
$$+ \cancelto{0}{\cdots}$$
$$= \int_{-\pi}^{\pi} a_n\cos(nx)\cos(nx)\,dx = a_n\cdot\pi$$

So, $\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx}$ cosine coefficients $n = 0, 1, 2, 3, \ldots$

repeating w/ $\sin(nx)$ ~~we~~ we see

$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx}$ sine coefficients $n = 1, 2, 3, \ldots$

try on $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ 0 & \text{if } 0 < x < \pi \end{cases}$ period 2π

calculate a_0 separately

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 1 \cdot dx = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \cos(nx)\,dx$$

$$= \frac{1}{n\pi}\sin(nx)\Big|_{-\pi}^{0} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} \sin(nx)\,dx$$

$$= -\frac{1}{n\pi}\cos(nx)\Big|_{-\pi}^{0} = -\frac{1}{n\pi}\left(1 - \cos(n\pi)\right)$$

$\downarrow$

-1 if n is odd

1 if n is even

$\Rightarrow (-1)^n$

$$= -\frac{1}{n\pi}\left(1-(-1)^n\right)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$